SOLUTION OF A SYSTEM OF HEAT AND MASS TRANSFER EQUATIONS BY THE METHOD OF STRAIGHT LINES

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An examination is made of the method of straight lines for solution of differential equations in partial derivatives relating to a system of heat and mass transfer equations.

In investigation of kinetics of a drying process there is a need to solve a linear system of heat and mass transfer equations [1]. The network method [2], for example, is applied with success to this problem and among other methods of solving equations in partial derivatives, the method of straight lines [3] is well known. A number of problems in heat transfer, mechanics, and hydrodynamics have been solved by this method [4–6]. The merit of the method is the fact that solution of the problem is reduced, eventually, to be a standard system of ordinary differential equations. For this reason, the method has an advantage compared with the network method when analog computers are used. Moreover, for linear systems, the method allows us to obtain analytical dependences of the desired functions on one of the independent variables, e.g., the Fourier number, for definite values of the other variable. It should be noted, however, that in the majority of cases of practical importance, a solution cannot be obtained in explicit form, and the method of straight lines encounters the same difficulty as the network method.

The essence of the straight lines method and some of its applications are described in detail in references [7–9, 13, 14].

We shall examine the linear system to which we may reduce the problem of heating and dehydration of a capillary-porous sphere in an external medium of constant temperature and with infinitely large velocity of propagation of heat and moisture:

$$\frac{\partial t}{\partial \mathrm{Fo}} = A_{11}\Delta t + A_{12}\Delta u,$$

$$\frac{\partial u}{\partial \mathrm{Fo}} = A_{21}\Delta t + A_{22}\Delta u, \tag{1}$$

where

$$A_{11} = 1 + \varepsilon\,\mathrm{KoLuPn},$$

$$A_{12} = \varepsilon\,\mathrm{KoLu},$$

$$A_{21} = \mathrm{LuPn},$$

$$A_{22} = \mathrm{Lu},$$

with the initial conditions

$$t(r,\ 0) = t_0,$$

$$u(r,\ 0) = u_0 \tag{2}$$

and boundary conditions of the 3rd kind

$$-\frac{\partial t}{\partial r}(1,\ \mathrm{Fo}) + \mathrm{Bi}_q[1 - t(1,\ \mathrm{Fo})] -$$

$$-(1-\varepsilon)\,\mathrm{KoLuBi}_m[u(1,\ \mathrm{Fo}) - u_e] = 0,$$

$$\frac{\partial u}{\partial r}(1,\ \mathrm{Fo}) + \mathrm{Pn}\frac{\partial t}{\partial r}(1,\ \mathrm{Fo}) + \mathrm{Bi}_m[u(1,\ \mathrm{Fo}) - u_e] = 0. \tag{3}$$

The dimensionless numbers are as follows:

$$\mathrm{Lu} = a_m/a_q,\ \mathrm{Bi}_q = \alpha R/\lambda,\ \mathrm{Bi}_m = \beta R/a_m,$$

$$\mathrm{Ko} = \rho u^0/c_q T_c^0,\ \ \mathrm{Pn} = \delta T_c^0/u^0,$$

$$\mathrm{Fo} = a_q \tau/R^2.$$

A condition of finiteness for the functions t and u is imposed, naturally, at the center of the sphere.

In accordance with the method we shall replace the derivatives with respect to r in (1) by the finite difference relations:

$$\frac{\partial t_k}{\partial r} \approx \frac{t_{k+1} - t_{k-1}}{2h};$$

$$\frac{\partial^2 t_k}{\partial r^2} \approx \frac{t_{k+1} - 2t_k + t_{k-1}}{h^2}.$$

Taking account of the fact that $r_k = kh$ and

$$(\Delta t)_k = \frac{\partial^2 t_k}{\partial r^2} + \frac{2}{r_k}\frac{\partial t_k}{\partial r},$$

we obtain

$$(\Delta t)_k \approx \frac{1}{kh^2}[(k+1)t_{k+1} - 2kt_k + (k-1)t_{k-1}],$$

$$(\Delta u)_k \approx \frac{1}{kh^2}[(k+1)u_{k+1} - 2ku_k + (k-1)u_{k-1}], \tag{4}$$

where h is the step size with regard to r (dimensionless); k = 1, 2, 3, ..., n; the number of internal straight lines is equal to n.

At the boundary (straight line with number k = n + + 1) we have:

$$(\Delta t)_{n+1} = \frac{\partial^2 t_{n+1}}{\partial r^2} + \frac{2}{r_{n+1}}\frac{\partial t_{n+1}}{\partial r},$$

$$t_n = t_{n+1} - h\frac{\partial t_{n+1}}{\partial r} + \frac{h^2}{2}\frac{\partial^2 t_{n+1}}{\partial r^2} - \dots,$$

$$\frac{\partial^2 t_{n+1}}{\partial r^2} \approx \frac{2}{h}\left(\frac{\partial t_{n+1}}{\partial r} - \frac{t_{n+1} - t_n}{h}\right),$$

$$(\Delta t)_{n+1} \approx \frac{2}{h}\left(\frac{\partial t_{n+1}}{\partial r} - \frac{t_{n+1} - t_n}{h}\right) + \frac{2}{r_{n+1}}\frac{\partial t_{n+1}}{\partial r}.$$

Table
Dependence of Temperature and Moisture Content at the Sphere Surface on the Fourier Number.

Fo	Temperature			Moisture Content		
	1 Line	3 Lines	6 Lines	1 Line	3 Lines	6 Lines
0	0	0	0	1	1	1
0.1	0.017	0.022	0.022	0.980	0.977	0.976
0.5	0.114	0.075	0.072	0.907	0.919	0.922
1	0.221	0.221	0.136	0.823	0.852	0.861
2	0.529	0.246	0.238	0.677	0.734	0.744
3	0.878	0.340	0.331	0.557	0.632	0.640
4	1.260	0.421	0.412	0.458	0.547	0.555

We write down $(\Delta u)_{n+1}$ at the boundary in similar fashion.

Allowing for the fact that $r_{n+1} = 1$, we obtain, for three straight lines, for example, the following system of ordinary differential equations:

$$t_1' = h^{-2}[A_{11}(2t_2 - 2t_1) + A_{12}(2u_2 - 2u_1)],$$

$$u_1' = h^{-2}[A_{21}(2t_2 - 2t_1) + A_{22}(2u_2 - 2u_1)],$$

$$t_2' = h^{-2}\left[A_{11}\left(\frac{3}{2}t_3 - 2t_2 + \frac{1}{2}t_1\right) + \right.$$

$$\left. + A_{12}\left(\frac{3}{2}u_3 - 2u_2 + \frac{1}{2}u_1\right)\right],$$

$$u_2' = h^{-2}\left[A_{21}\left(\frac{3}{2}t_3 - 2t_2 + \frac{1}{2}t_1\right) + \right.$$

$$\left. + A_{22}\left(\frac{3}{2}u_3 - 2u_2 + \frac{1}{2}u_1\right)\right],$$

$$t_3' = h^{-2}\left[A_{11}\left(\frac{4}{3}t_4 - 2t_3 + \frac{2}{3}t_2\right) + \right.$$

$$\left. + A_{12}\left(\frac{4}{3}u_4 - 2u_3 + \frac{2}{3}u_2\right)\right],$$

$$u_3' = h^{-2}\left[A_{21}\left(\frac{4}{3}t_4 - 2t_3 + \frac{2}{3}t_2\right) + \right.$$

$$\left. + A_{22}\left(\frac{4}{3}u_4 - 2u_3 + \frac{2}{3}u_2\right)\right],$$

$$t_4' = A_{11}(\Delta t)_4 + A_{12}(\Delta u)_4,$$

$$u_4' = A_{21}(\Delta t)_4 + A_{22}(\Delta u)_4, \qquad (5)$$

where

$$(\Delta t)_4 = 2\frac{\partial t}{\partial r}(1, \mathrm{Fo}) + \frac{2}{h}\left[\frac{\partial t}{\partial r}(1, \mathrm{Fo}) - \frac{t_4 - t_3}{h}\right],$$

$$(\Delta u)_4 = 2\frac{\partial u}{\partial r}(1, \mathrm{Fo}) + \frac{2}{h}\left[\frac{\partial u}{\partial r}(1, \mathrm{Fo}) - \frac{u_4 - u_3}{h}\right].$$

The initial conditions remain as before, while the derivatives with respect to functions t and u on the sphere surface are substituted from the boundary conditions (3) into the system (5). Thus the boundary conditions are included in the system in the form of differential equations. However, the order of the system may be lowered by two, if the boundary conditions are considered in the form of algebraic equations, by replacing derivatives of the first order with respect to r by finite difference relations. This may prove to be important when working with a small number of straight lines.

The convergence of the method of straight lines for linear systems, and also for systems with variable coefficients in a rectangular region follows from the lemma of section 3 of [10], under the following sufficient condition:

$$\frac{\partial \varphi}{\partial t} + \left|\frac{\partial \varphi}{\partial u}\right| \leqslant \mathrm{const} < 0,$$

$$\frac{\partial \psi}{\partial u} + \left|\frac{\partial \psi}{\partial t}\right| \leqslant \mathrm{const} < 0, \qquad (6)$$

where φ and ψ are the right sides in the boundary condition of the 3rd kind:

$$\frac{\partial t}{\partial r}(R, \tau) = \varphi[\tau, t(R, \tau), u(R, \tau)],$$

$$\frac{\partial u}{\partial r}(R, \tau) = \psi[\tau, t(R, \tau), u(R, \tau)].$$

Here we assume the existence and uniqueness of the solution of the mixed problem (1)–(3), and the continuity and sufficient smoothness of the coefficients and of the solutions themselves.

In the case of the linear system (1) the condition of convergence has the form

$$\mathrm{Bi}_q > (1 - \varepsilon)\mathrm{KoLuBi}_m,$$

$$\mathrm{Bi}_m > (1 - \varepsilon)\mathrm{PnKoLuBi}_m + \mathrm{Bi}_q\mathrm{Pn}.$$

The estimates demonstrating the convergence of the method are certainly above the accuracy of interest to the investigator, as a rule. We therefore require to solve the problem for 1, 2, 3, etc., straight lines, with the object of determining the relative discrepancy between the respective approximations. A tentative criterion of convergence of the method in a given region of variation of the arguments could be a monotonic decrease in these discrepancies as we increase the number of straight lines taken. A measure of the accuracy are the discrepancies, required especially in solution of non-linear problems, in which it is not possible to compare the approximate results with an exact analytical expression. It should be noted that the above empirical criterion for assessing the accuracy of the method is the same as in the network method, when one wishes to obtain the "optimum" step sizes to secure the required accuracy for a minimum expenditure of effort [11].

We shall solve the system (5) and obtain the desired functions of Fo on the given straight lines. Carrying out the appropriate numerical integration with respect to r, we find the dependence of the mean values of temperature and moisture content on Fo. In the event that the deviation of the central functions from the surface functions is not appreciable, we may use only the surface functions in investigating the technological conditions.

To illustrate the above method we shall examine a numerical example with the following initial data: $Bi_q = 0.05$; $Bi_m = 0.05$; Lu = 1; Ko = 0.1; Pn = 0; $\varepsilon = 1$; $u_e = 0$; $t_0 = 0$; $u_0 = 1$. The results of the calculation of temperature and moisture content on the sphere surface are presented in the Table.

It may be seen from the Table how sharply the values of the desired functions differ for one and three lines, whereas the deviations between the values for three lines and six lines are already no greater than 5%. A similar thing is true of the dependence of temperature and of moisture content on Fo on the internal straight lines. The dependence of the above quantities on the coordinate r has the form of a parabola whose curvature drops with increase of Fo, which corresponds to the description of the actual process.

The specific manner of applying the method is the same for a non-linear system as for the linear case, the coefficients being the unknown functions t_k and u_k, while the system of ordinary differential equations remains non-linear. Examples of a non-linear finite difference exercise of the method are the cases dealt with in [2], where the solution is obtained of a mixed problem for a coupled transfer system by the network method, and in [12], where an unsteady heat transfer problem is solved on analog computers by the method of straight lines.

By way of comment on the insufficient study that the method has received, we note the possibility of using it to solve a number of particular problems in heat and mass transfer.

NOTATION

t is the temperature of the body being examined, relative to the temperature of the medium T_M^0; u is the specific moisture content of the body under examination, relative to the initial moisture content u^0; r is the space coordinate, referred to the sphere radius R; u_e is the equilibrium moisture content, relative to the initial moisture content u^0; Fo, Lu, Ko, Pn, Bi_q, Bi_m are the Fourier, Lykov, Kossovich, Posnov, and heat and mass transfer Biot numbers; a_m is the moisture diffusion coefficient; a_q is the thermal diffusivity; λ is the thermal conductivity; c_q is the specific heat of the moist body; δ is the thermogradient coefficient; α is the heat transfer coefficient; β is the mass transfer coefficient; ε is the phase transformation parameter; τ is the time; Δ is the Laplace operator; k is the number of the straight line; t', u' are the derivatives of the desired functions with respect to Fo; $\partial t_k/\partial r$ is the derivative at the point (r_k, Fo).

REFERENCES

1. A. V. Luikov, Transfer Phenomena in Capillary-Porous Substances [in Russian], Gostekhizdat, 1954.
2. R. I. Gavrilova, IFZh, 710, 8, 1964.
3. M. G. Slolodyanskii, Prikl. matem. i mekh., 3, no. 1, 1939.
4. P. E. Burtsev, Tr. Mosk. energet, in-ta, no. 32, 1959.
5. N. I. Chelnokov, Tr. Mosk. energet, in-ta, no. 41, 1962.
6. G. M. Komladze, Tr. Tbil. gos. ped. in-ta, 14, 1959.
7. V. N. Faddeeva, Tr. Matem. in-ta AS USSR im. V. A. Steklova, 28, 1949.
8. I. S. Berezin and N. P. Zhidkov, Computational Methods [in Russian], Fizmatgiz, 2, 1962.
9. I. I. Etterman, Continuous Mathematical Computers [in Russian], Mashgiz, 1957.
10. B. M. Budak, Zhurnal vych. matem. i mat. fiz., 1, no. 6, 1961.
11. V. K. Saul'ev, Integration of Equations of Parabolic Type by the Network Method [in Russian], Fizmatgiz, 1960.
12. L. I. Kudryashev and A. V. Temnikov, Heat and Mass Transfer, 5 [in Russian], Izd. AS BSSR, 1963.
13. O. A. Liskovets, Cand. Thesis of the BGU [in Russian], Minsks, 1964.
14. O. A. Liskovets, Differentsial'nye uravneniya [Differential Equations], 1, no. 12, 1965.

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